Quantisation of Bending Flows

Gregorio Falqui
Dipartimento di Matematica e Applicazioni, Università di Milano–Bicocca, via R. Cozzi 53, 20125 Milano, Italy, gregorio.falqui@unimib.it
Fabio Musso
Dipartimento di Fisica, Università di Roma III, and INFN, Sezione di Roma III, via Vasca Navale, 84, 00146 Roma, Italy, musso@fis.uniroma3.it

Abstract

We briefly review the Kapovich-Millson notion of Bending flows as an integrable system on the space of polygons in $\mathbf{R}^3$, its connection with a specific Gaudin XXX system, as well as the generalisation to $su(r), r > 2$. Then we consider the quantisation problem of the set of Hamiltonians pertaining to the problem, quite naturally called Bending Hamiltonians, and prove that their commutativity is preserved at the quantum level.

1 Introduction: Classical Bending Flows and Gaudin Models

Bending flows were introduced by Kapovich and Millson (KM) in [1], as particular Hamiltonian integrable systems on the manifold of "moduli" of n-sided polygons (n–gons) in $\mathbf{R}^3$. They can be briefly described as follows.

An n-gon in Euclidean 3-space is specified by its n vertices $\{v_1, \ldots, v_n\}$, or, up to a translation, by its n sides $\{e_1, \ldots, e_n\}$, the lengths of the elements of the latter being given by a suitable string $\mathbf{r} = \{r_1, \ldots, r_n\}$ of non-negative numbers. The moduli space of such n-gons, $\mathcal{M}_{n,\mathbf{r}}$, is obtained by factoring out the action of the Euclidean group on the set of n-gons with preassigned side–lengths. It can be identified with the Hamiltonian reduction of the Cartesian product $P_{n,\mathbf{r}}$ of n two-spheres $\mathbf{S}_i^2$ of radius r_i (endowed with the standard symplectic two-form), with respect to the Hamiltonian action of the group $SO(3)$, the closure condition of the n-gon being translated in the fact that the value of the moment map to be considered is the zero value. Thus the *moduli space* of n-gons in $\mathbf{R}^3$, $\mathcal{M}_{n,\mathbf{r}} = \mathcal{P}_{n,\mathbf{r}}//_{SO(3)}$, is a $2n-6$ dimensional symplectic manifold.

A completely integrable classical Hamiltonian system on $\mathcal{M}_{n,\mathbf{r}}$ can be defined ([1]) on this phase space as follows. Fix one of the vertices of a polygon, say v_1, and consider the $n-3$ diagonals $\{d_1, \ldots, d_{n-3}\}$ stemming from v_1, and their squared lengths $h_\alpha = ||d_\alpha||^2$. It can be proven that the functions h_α are mutually in involution and functionally independent on $\mathcal{M}_{n,\mathbf{r}}$, and thus give rise to a Liouville integrable system. The name *Bending flows* for these Hamiltonian

systems comes from the fact that the motion induced by the Hamiltonians h_α is a rotation (with constant velocity) of the sub-polygon defined by the edges $\{e_1, e_2, \dots, e_{\alpha+1}, d_\alpha\}$ around the diagonal d_α, while the complementary sub-polygon $\{d_\alpha, e_{\alpha+2}, \dots, e_n\}$ is left fixed in $\mathbf{R}^3$. In other words, "half" of the n–gon *bends* around the α-th diagonal.

In [2] a connection of these flows with (a particular case of) XXX classical Gaudin model was described. Indeed, the KM moduli space coincides with a subset of the phase space of such $su(2)$ Gaudin magnet with $N = n - 3$ sites, and the KM Hamiltonians h_α were shown to be the classical analogue of a set of mutually commuting operators found by Ballesteros, Ragnisco and collaborators (see, e.g., [3]) in their study of Hopf algebraic properties of quantum Gaudin magnets.

The classical limits of these Hamiltonians are represented as follows. It is known that the classical Gaudin system admits a Lax representation, with Lax matrix

$$L(\lambda) = \sum_{i=1}^{N} \frac{A_i}{\lambda - z_i}, \tag{1}$$

the z_i being a set of numerical parameters, while the A_i being $su(2)$-valued matrices that encode the degrees of freedom of the model. The Hamiltonians introduced by Ballesteros and Ragnisco, that we herewith consider are given by

$$J_\alpha = \frac{1}{2}\mathrm{T}r \left(\sum_{i=1}^{\alpha+1} A_i \right)^2, \quad \alpha = 1, \dots, N-1. \tag{2}$$

Together with a component of the total spin these J_a's provide a set of mutually commuting integrals of the motion for the "physical" Gaudin Hamiltonian

$$H_G = \frac{1}{2} \sum_{i \neq j = 1}^{N} \mathrm{Tr}(A_i A_j), \tag{3}$$

that is, alternative to the "standard" one, namely, the set of spectral invariants of the Lax matrix (1), i.e.:

$$H_\alpha = \mathrm{Res}_{\lambda = z_\alpha} \mathrm{Tr} L(\lambda)^2. \tag{4}$$

This identification suggests the opportunity to study, among others, the following three problems:

A) Generalise bending flows to the $su(n)$ case, and to a general Lie algebra $\mathfrak{g}$.

B) Use the identification to provide "new" ways of solving the classical counterparts of the Gaudin models encompassed by this framework.

C) Tackle the quantum case (with $\mathfrak{g}$-valued spins) from this standpoint.

Points A and B above were described and substantially solved in [2] and [4]. The main idea was to use tools from the bihamiltonian theory of integrable systems. The basic point is that the phase space of $\mathfrak{g}$-valued Gaudin models is, as it is well known, a symplectic submanifold (*a leaf*) in the tensor product $(\mathfrak{g}^*)^{\otimes N}$ of N copies of the dual of the Lie algebra $\mathfrak{g}$, endowed with the standard Lie–Poisson brackets. A suitable additional Poisson structure was defined on $(\mathfrak{g}^*)^{\otimes N}$ in [2] with the following properties:
1) The Poisson brackets defined by this new structure are compatible (in the Magri sense) with the Lie-Poisson brackets, so that they define, on $(\mathfrak{g}^*)^{\otimes N}$, a *pencil* of Poisson brackets.
2) The Lenard-Magri sequences defined by this pencil give rise, in the $\mathfrak{g} = su(2)$ case, to the KM integrals (2); in the case of general simple $\mathfrak{g}$, the bihamiltonian iteration provides a set of (bi–involutive) integrals of the motion; they were called (*generalised*) Bending Hamiltonians. These Hamiltonians, when complemented with a suitably chosen set of integrals associated with the global invariance under the Lie group $G = \exp(\mathfrak{g})$, insure the Liouville integrability of the model.

An instrumental feature of this scheme is the following. The integrals of the motion associated with point 2) of the above list, can be obtained as follows. For $a = 2, 3, \dots N$ one can introduce $\mathfrak{g}$-valued matrices

$$L_a(\lambda) = (\lambda - (a-2))A_\alpha + \sum_{i=1}^{a-1} A_i. \tag{5}$$

These matrices are Lax matrices for the Hamiltonian flows associated with the generalised Bending Hamiltonians, that is, they evolve isospectrally along the generalised Bending flows; the ring of their spectral invariants coincides with that of the generalised Bending Hamiltonians.

Furthermore, these Lax matrices provide, according to the so-called Sklyanin magic recipe [5] and the bihamiltonian scheme of Separation of Variables (see, e.g., [6]), a set of separation coordinates. It can be noticed [2] that, in the case of $\mathfrak{g} = sl(r)$, the solution to the HJ equations involves integration of Abelian differentials on Riemann surfaces of genus $g = (r-1)(r-2)/2$, irrespectively of the number N of sites (while the SoV scheme associated with the "single" Lax matrix (1) involves a Riemann surface whose genus grows linearly with N).

What is more important for the purpose of the present paper is that they will be the basic objects for the quantisation of the Bending Hamiltonians (point C of the problem list of the previous page), to be discussed in the sequel.

2 Quantisation of the bending Hamiltonians

The problem of the quantisation of Gaudin Hamiltonians (with $\mathfrak{g}$-valued spins) was discussed (see, e.g., [7] and the references quoted therein), via the quantisation of the spectral invariants of the "Sklyanin" Lax matrix (1), and their diagonalisation via Bethe Ansatz techniques. Here we will consider the spectral invariants associated with the Lax matrices of the Bending Flows(5).

As in the usual case, the matrices A_i (and hence the Bending Lax matrices L_a) will become operators in a suitable Hilbert space, which, in the present paper, will be the N-th tensor product of a faithful representation ρ of a simple Lie algebra $\mathfrak{g}$. Their quantisation is given by the following recipe:

$$A_i = g_{\alpha\beta}\rho(X^\alpha)y_i^\beta \ \rightarrow \ A_i^{(q)} = g_{\alpha\beta}\rho(X^\alpha)X_i^\beta \tag{6}$$

$$L_a = \lambda A_a + \sum_{k=1}^{a-1}(A_k) \ \rightarrow \ L_a^{(q)} = \lambda A_a^{(q)} + \sum_{k=1}^{a-1}(A_k^{(q)}) \tag{7}$$

Here, $g_{\alpha\beta}$ are the components of the inverse of the Cartan matrix of $\mathfrak{g}$ and we shifted the spectral parameter appearing in (5). In [3] it was already noticed that the straightforward quantisation of the specific Hamiltonians $\mathrm{Tr}\big((\sum_{k=1}^{j} A_k)^i\big)$ do commute (without any quantum correction). However, the number of such Hamiltonians grows linearly with the number N of sites of the magnet, and so cannot provide a complete set of commuting operators for generic representations ρ of $\mathfrak{g}$ whenever $\mathfrak{g} \neq su(2)$.

In this section we will extend this result to a larger set of generalised bending Hamiltonians, using in the following the $i-$coproducts in the universal enveloping algebra, $\Delta^{(i)} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\otimes i}$:

$$\Delta^{(i)}(1) = 1, \quad \Delta^{(i)}(X^\alpha) = \sum_{k=1}^{i} X_k^\alpha, \ \ X^\alpha \in \mathfrak{g}, \quad \Delta^{(i)}(ab) = \Delta^{(i)}(a)\Delta^{(i)}(b)\,. \tag{8}$$

We will make use of the following identity:

$$\left[\mathrm{Tr}\left(L_a^{(q)}(\lambda)^m\right), \Delta^{(p)}(X^\alpha)\right] = 0 \quad \text{for } p > a \quad X^\alpha \in \mathfrak{g}, \tag{9}$$

whose proof goes as follows.

¿From the fact that $\mathrm{Tr}(A_i^m)$ give us central elements of $U(\mathfrak{g})$ we know that, for $X^\alpha \in \mathfrak{g}$, and every index i, it holds that $[\mathrm{Tr}(A_i^m), X_i^\alpha] = 0$. Defining:

$$g_{j_1 \ldots j_m} := g_{i_1 j_1} g_{i_2 j_2} \ldots g_{i_m j_m} \mathrm{Tr}\left(\rho(X^{i_1}) \ldots \rho(X^{i_m})\right) \tag{10}$$

one can obtain the identity $\sum_{r=1}^{m} g_{j_1 \ldots j_{r-1} s j_{r+1} \ldots j_m} C_{j_r}^{sq} = 0$ for every $j_1, \ldots, j_m$. Using this and the commutation relations ($\Delta^{(i)}$ is a Lie algebra homomorphism)

$$\left[X_{i+1}^j, \Delta^{(p)}(X^q)\right] = C_s^{jq} X_{i+1}^s, \qquad \left[\Delta^{(i)}(X^j), \Delta^{(p)}(X^q)\right] = C_s^{jq}\Delta^{(i)}(X^s) \quad p > i\,,$$

we can conclude

$$\left[\mathrm{Tr}\left((L_i^{(q)}(\lambda))^m\right), \Delta^{(p)}(X^q)\right] = \left[\mathrm{Tr}\left(\left(\lambda A_{i+1} + \Delta^{(i)}(A)\right)^m\right), \Delta^{(p)}(X^q)\right] =$$
$$= \left[g_{j_1\dots j_m}(\lambda X_{i+1}^{j_1} + \Delta^{(i)}(X^{j_1})\dots(\lambda X_{i+1}^{j_m} + \Delta^{(i)}(X^{j_m}), \Delta^{(p)}(X^q)\right] =$$
$$\sum_{r=1}^{m} g_{j_1\dots j_{r-1}sj_{r+1}\dots j_m} C_{j_r}^{sq}((\lambda X_{i+1}^{j_1} + \Delta^{(i)}(X^{j_1}))\dots(\lambda X_{i+1}^{j_m} + \Delta^{(i)}(X^{j_m})) = 0.$$

Eq. (9) is central in our analysis; indeed it entails that

$$\left[\mathrm{Tr}\left(L_i^{(q)}(\lambda)^m\right), \mathrm{Tr}\left(L_j^{(q)}(\mu)^n\right)\right] = 0 \qquad \text{if } i \neq j \quad \forall m, n, \tag{11}$$

for arbitrary values of the spectral parameters λ, μ. In fact, assuming $j > i$, we have

$$\mathrm{Tr}\left(L_j^{(q)}(\mu)^n\right) = g_{i_1\dots i_n}(\mu X_{j+1}^{i_1} + \Delta^{(j)}(X^{i_1}))\dots(\mu X_{j+1}^{i_n} + \Delta^{(j)}(X^{i_n})). \tag{12}$$

Since $[\mathrm{Tr}\left(L_i^{(q)}(\lambda)^m\right), \Delta^{(j)}(X^q))] = 0$, and $[\mathrm{Tr}\left(L_i^{(q)}(\lambda)^m\right), X_{j+1}^q] = 0$, the vanishing commutation relations (11) are actually verified.

Let us now consider the quantum (Bending) Hamiltonians as defined via

$$H_{l,m}^{(a)} := \mathrm{res}_{\lambda=0}\frac{1}{\lambda^{l+1}}\mathrm{Tr}\left(L_a^{(q)}(\lambda)^m\right), \tag{13}$$

where $a = 2, \dots, N,\ l = 0, \dots, m$ and m runs in the set of the exponents of $\mathfrak{g}$. We notice that the set of Hamiltonians (13) contains those defined in [3], since $\mathrm{Tr}((\sum_{k=1}^{a} A_k^q)^m) = H_{0,m}^{(a)}$; also, they satisfy the following equality:

$$\sum_{l=0}^{m} H_{l,m}^{(a)} = H_{0,m}^{(a+1)} \qquad a = 2, \dots, N \tag{14}$$

where, in the case $a = N$ this equation defines $H_{0,m}^{(N+1)}$.

With our definitions, Eq. 9 in particular entails that Hamiltonians coming from two different Lax matrices of the family (5) quantum commute, i.e.,

$$[H_{l,m}^{(a)}, H_{p,q}^{(b)}] = 0 \quad \forall p, q, l, m \qquad \text{if } a \neq b\,. \tag{15}$$

We are thus left to consider the commutators among Hamiltonians of the family (13) coming from the same Lax matrix. We deem that using (possibly with suitable modifications) the techniques discussed in the recent papers [8, 9] based on the notion of *quantum determinant*, the commutativity of such quantities can be proven in full generality. However, in the last part of the present paper, we shall show that coproduct methods provide a complete answer for cubic Hamiltonians (and hence, for $\mathfrak{g} = su(3)$). Namely, we can show that specific

Hamiltonians (or suitable linear combinations thereof) up to the third order provide commuting elements; we notice that no quantum corrections to the straightforward quantisation procedure are required.

To do so, we first show that it holds, still for any value of the indexes,

$$[H^{(a)}_{m,m}, H^{(b)}_{p,q}] = 0 \qquad \text{and} \qquad [H^{(a)}_{0,m}, H^{(b)}_{p,q}] = 0. \tag{16}$$

The first of these identities follows from the fact that $H^{(a)}_{m,m} = \mathrm{Tr}\,((A_a)^m)$, for $a = 2, \ldots, N$, are Casimirs elements of $U(\mathfrak{g})^{\otimes N}$.

Regarding the second of (16), we know that the Hamiltonians will commute if $a \neq b$, so we have to consider just the case $a = b$. In this case we can use equations (14,15) to write:

$$[H^{(a)}_{0,m}, H^{(a)}_{p,q}] = \sum_{l=0}^{m} [H^{(a-1)}_{l,m}, H^{(a)}_{p,q}] = 0 \qquad a > 2. \tag{17}$$

On the other hand, if $a = 2$, then $H^{(2)}_{0,m}$ is a Casimir and (16) is verified as well.

Our final task is to prove that

$$\left[\mathrm{Tr}\left(L^{(q)}_a(\lambda)^m\right), \mathrm{Tr}\left(L^{(q)}_b(\mu)^n\right)\right] = 0 \qquad \text{if} \quad m, n = 1, 2, 3, \quad \forall\, a, b. \tag{18}$$

To our end, we use the explicit expressions

$$\mathrm{Tr}\left(L^{(q)}_a(\lambda)^2\right) = \lambda^2 H^{(i)}_{2,2} + \lambda H^{(a)}_{1,2} + H^{(a)}_{0,2} \tag{19}$$

$$\mathrm{Tr}\left(L^{(q)}_a(\lambda)^3\right) = \lambda^3 H^{(a)}_{3,3} + \lambda^2 H^{(a)}_{2,3} + \lambda H^{(a)}_{1,3} + H^{(a)}_{0,3}. \tag{20}$$

We want to show that all the seven Hamiltonians appearing here commute among themselves. Thanks to the result (16), we are left to prove the commutativity of the three quantities $H^{(a)}_{1,2}$, $H^{(a)}_{1,3}$, $H^{(a)}_{2,3}$. We have:

$$[H^{(a)}_{1,2}, H^{(a)}_{1,3}] = [H^{(a)}_{0,2} + H^{(a)}_{1,2} + H^{(a)}_{2,2}, H^{(a)}_{1,3}] = [H^{(a+1)}_{0,2}, H^{(i)}_{1,3}] = 0 \tag{21}$$

where we used the fact that $[H^{(a)}_{0,2}, H^{(a)}_{1,3}] = [H^{(a)}_{2,2}, H^{(a)}_{1,3}] = 0$, equation (14) and equation (15).

Since Eqn's $[H^{(a)}_{1,2}, H^{(a)}_{2,3}] = 0$ and $[H^{(a)}_{1,3}, H^{(a)}_{2,3}] = 0$ can be proven exactly in the same way, we have thus shown the commutativity of a complete set of the quantised Bending Hamiltonians (13) in the case $\mathfrak{g} = su(3)$.

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